

Note About Unstable D-Brane with Dynamical Tension

J. Klusoň,^{a, 1}

^a*Department of Theoretical Physics and Astrophysics, Faculty of Science,
Masaryk University, Kotlářská 2, 611 37, Brno, Czech Republic*

Abstract

We propose an action for unstable Dp-brane with dynamical tension. We show that the equations of motion are equivalent to the equations of motion derived from DBI and WZ actions for non-BPS Dp-brane. We also find Hamiltonian formulation of this action and analyze properties of the solutions corresponding to the tachyon vacuum and zero tension solution.

¹Email address: klu@physics.muni.cz ,

1 Introduction and Summary

It is well known that string theories are not only theories of strings but also contain number of objects with different dimensionality and properties ². D-branes are examples of these objects that are in some sense exceptional since they can be exactly defined "as the planes where open strings can end" and hence they have exact two dimensional conformal field theory description at least at the weak coupling regime [7]. Characteristic properties of these objects (among others) is an existence of the constant known as p -brane tension T_p which is the mass per unit of spatial p -dimensional volume. The value of the tensions for different objects can be determined by different methods, see for example excellent book [3]. On the other hand Dirac-Born-Infeld (DBI) action for Dp -brane is highly non-linear due to the square root of the determinant. Further, this action cannot describe objects with zero tension. This problem can be solved by introducing auxiliary fields so that we replace square-root structure of the action with the more tractable one when we introduce scalar auxiliary field. Then we can go even further and replace the constant T_p by p -form gauge potential in such a way that the tension arises as solution of the equations of motion for this non-dynamical p -form [4, 5, 6]. This is very attractive form of the action that is also scale invariant. It is also important to stress that the integration constant can have any value so that it describes tensionless and negative tension branes [8] as well, at least in principle. On the other hand the case of the zero integration constant is again tricky as the tensionless limit of ordinary Dirac-Born-Infeld action since it implies that the matrix is singular and hence the derivation of the equation of motion that is valid in case of the non-zero tension cannot be applied here. The correct procedure how to analyze zero tension solution is to switch to the Hamiltonian formalism.

Since the idea of the Dp -brane with dynamical tension is very attractive we can try to extend this construction to the case of an unstable non-BPS Dp -brane [9, 10, 11, 12] ³ which is the goal of this paper. We propose an action for non-BPS Dp -brane with variable tension. Then we determine corresponding equations of motion. We show that the tachyon kink is the solution of these equations of motion on condition that the solution of the equation of motion for p -form is non-zero constant. Then we argue, following [14, 15] that this kink on the world-volume of unstable Dp -brane corresponds to the $D(p-1)$ -brane. It is clear that the equations of motion and kink solution are not valid in case when the solution of the equation of motion for p -form is zero constant. In order to analyze this problem we proceed to the Hamiltonian formulation of a non-BPS Dp -brane with variable tension. We find corresponding Hamiltonian and determine algebra of constraints. Then we analyze two situations. The first one when the tachyon is sitting at its minimum value. We argue that the resulting equations of motion with constant electric flux correspond to the equations of motion derived from the Nambu-Gotto action. In other words the dynamics of the non-BPS Dp -brane at the tachyon vacuum is equivalent to the

²For recent review, see [1, 2, 3].

³For review, see [13].

dynamics of fundamental string that is delocalized along the world-volume of non-BPS Dp-brane ⁴. We mean that the fact that this solution is delocalized along the world-volume of an unstable D-brane has a natural interpretation. We know that at the end of the tachyon vacuum unstable D-brane disappears so that we mean that it does not make sense to speak about the position of the string remnant on it. Further, the only physical meaning has the localization of the string in the target space-time and the dynamics of these modes is governed by the equations of motion that follow from Nambu-Gotto action. We also discuss the second class of the solution of the Hamiltonian equations of motion for non-BPS Dp-brane with variable tension when the tension of this brane is equal to zero. The similar situation was analyzed previously in [21, 22, 23, 24] however with slightly different limiting procedure. More precisely, the limiting procedure introduced in [21, 22, 24, 23] the factor in front of $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ scales as well which is not the case of the non-BPS Dp-brane with the variable tension where the gauge field has the dimension of length. This follows from the fact that the tension is generated dynamically as the solution of the equations of motion. On the other hand there is no way how to generate length scale in front $F_{\mu\nu}$ dynamically. As a result the zero tension solution of the non-BPS Dp-brane equation of motion has similar form as the tachyon vacuum solution with the difference that now the world-volume theory contains additional massless mode which was the original tachyon field. In other words the solutions of the equations motion at the zero tension vacuum correspond to the string propagating in the space with one additional dimension.

All these results are derived in the background with vanishing Ramond-Ramond (RR) fields so that one can ask the question whether an existence of the background with non-trivial RR fields does not change the interpretation of the resulting solutions as the solutions that arise from the Nambu-Gotto action. In other words, if we were found that there is non-zero coupling to the RR fields at the tachyon vacuum we could not interpret the resulting configuration as the fundamental string due to the fact that fundamental string does not couple to RR fields directly. We demonstrate that this is not the case on the example of non-BPS D2-brane when we find its Hamiltonian formulation in the presence of the RR fields. We show that the coupling of this brane to RR fields vanishes at the tachyon vacuum and hence the resulting configuration really corresponds to the fundamental string. This is non-trivial result since the Hamiltonian formulation of non-BPS D2-brane in the background with non-zero RR fields has not been done before.

Let us outline our results. We propose an action for non-BPS Dp-brane with dynamical tension. We study its properties and show that the equations of motion for this system has the tachyon kink solution that can be interpreted as a lower dimensional D(p-1)-brane. We also find the Hamiltonian formulation of p-brane and non-BPS Dp-brane with variable tension in general background. We study two solutions that cannot be analyzed in the conventional Lagrangian formalism which are the tachyon vacuum solution and the zero tension solution. We argue that

⁴For previous works where the fate of the non-BPS Dp-brane at the tachyon vacuum was analyzed, see [16, 17, 18, 19, 20].

the solutions of the equations of motion at these vacua and with constant electric flux have a natural interpretation as the solutions of the Nambu-Gotto equations of motion which supports the conjecture that at the tachyon vacuum the non-BPS Dp-brane disappears and the gas of the fundamental string emerges. Finally we also argue that this conclusion is valid even in the presence of the non-trivial Ramond-Ramond forms.

The organization of this paper is as follows. In the next section (2) we introduce an action for Dp-brane in general background with dynamical tension and we show that the resulting equations of motion are equivalent to the equations of motion derived from DBI and WZ D-brane actions. In section (3) we perform Hamiltonian analysis of p-brane with the variable tension. In section (4) we formulate an action for non-BPS Dp-brane with variable tension and we show that the equations of motion have a solution that can be interpreted as D(p-1)-brane. In section (5) we find Hamiltonian formulation of this Dp-brane and analyze canonical equations of motion. We also find the Hamiltonian for non-BPS D2-brane with dynamical tension in non-zero RR background.

2 D-brane Action With Variable Tension

In this section we review basic facts about Dp-brane with variable tension following [6]. Let us consider the Lagrangian density in the form

$$\mathcal{L} = \frac{1}{2v} [e^{-2\Phi} \det \mathbf{A} + (\star \mathcal{G}_{(p+1)})^2] , \quad (1)$$

where Φ is a dilaton, v is an independent worldvolume density, and where

$$\mathbf{A}_{\mu\nu} = g_{\mu\nu} + \mathcal{F}_{\mu\nu} , \quad \mathcal{F}_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + b_{\mu\nu} , \quad (2)$$

where $g_{\mu\nu}$ and $B_{\mu\nu}$ are induced metric and two form respectively

$$g_{\mu\nu} = G_{MN} \partial_\mu X^M \partial_\nu X^N , \quad b_{\mu\nu} = B_{MN} \partial_\mu X^M \partial_\nu X^N . \quad (3)$$

G_{MN} and B_{MN} , $M, N = 0, \dots, 9$ are target space-time metric and NS-NS two form fields respectively, $X^M(\xi)$ are world-volume fields that parameterize the position of Dp-brane in the target space-time. Finally $V_\mu, \mu = 0, \dots, p$ is world-volume gauge field where the world-volume of Dp-brane is labeled with coordinates $\xi^\mu, \mu = 0, \dots, p$.

An important building block of Dp-brane with the variable tension is the scalar density $\star \mathcal{G}_{(p+1)}$ which is the world-volume Hodge dual of $(p+1)$ -form field strength $\mathcal{G}_{(p+1)}$ and that has an explicit form

$$\star \mathcal{G}_{(p+1)} = \frac{1}{(p+1)!} \epsilon^{\mu_1 \dots \mu_{p+1}} \partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]} - \sum_{n \geq 0} \frac{1}{n!(2!)^n q!} \epsilon^{\mu_1 \dots \mu_{p+1}} (\mathcal{F})_{\mu_1 \dots \mu_{2n}}^n C_{\mu_{2n+1} \dots \mu_{p+1}} , \quad (4)$$

where $q = p + 1 - 2n$ and where

$$C_{\mu_1 \dots \mu_p} = C_{M_1 \dots M_p} \partial_{\mu_1} X^{M_1} \dots \partial_{\mu_p} X^{M_p} \quad (5)$$

is the pull-back of the Ramond-Ramond forms to the world-volume of Dp-brane.

Our goal is to show that the equations of motion that follow from the action $S = \int d^{p+1} \xi \mathcal{L}$ are equivalent to the equations of motion that follow from the DBI and WZ actions for ordinary Dp-brane.

To begin with note that the equation of motion for A implies that

$$\star \mathcal{G}_{(p+1)} = T v , \quad (6)$$

where T is a constant. In what follows we will presume that it is not equal to zero. On the other hand the equation of motion for v implies

$$\star \mathcal{G}_{(p+1)} = e^{-\Phi} \sqrt{-\det \mathbf{A}} \quad (7)$$

that together with (6) implies

$$v = \frac{1}{T} e^{-\Phi} \sqrt{-\det \mathbf{A}} . \quad (8)$$

Finally we analyze the equations of motion for X^M and V_α . In case of X^M we obtain

$$\begin{aligned} & \frac{1}{2v} \partial_M [e^{-2\Phi}] \det \mathbf{A} + \frac{1}{2v} e^{-2\Phi} (\partial_M G_{KL} \partial_\alpha X^K \partial_\beta X^L + \partial_M B_{KL} \partial_\alpha X^K \partial_\beta X^L) (\mathbf{A}^{-1})^{\beta\alpha} \det \mathbf{A} \\ & - \partial_\alpha \left[\frac{1}{v} e^{-2\Phi} G_{MN} \partial_\beta X^N (\mathbf{A}^{-1})_S^{\beta\alpha} \det \mathbf{A} \right] - \partial_\alpha \left[\frac{1}{v} e^{-2\Phi} B_{MN} \partial_\beta X^N (\mathbf{A}^{-1})_A^{\beta\alpha} \det \mathbf{A} \right] + J_M = 0 , \end{aligned} \quad (9)$$

where

$$J_M = \int d^{p+1} \xi \frac{1}{v} \frac{\delta(\star \mathcal{G}_{(p+1)})}{\delta X^M} \star \mathcal{G}_{(p+1)} , \quad (10)$$

and where

$$(\mathbf{A}^{-1})_S^{\alpha\beta} = \frac{1}{2} ((\mathbf{A}^{-1})^{\alpha\beta} + (\mathbf{A}^{-1})^{\beta\alpha}) , \quad (\mathbf{A}^{-1})_A^{\alpha\beta} = \frac{1}{2} ((\mathbf{A}^{-1})^{\alpha\beta} - (\mathbf{A}^{-1})^{\beta\alpha}) . \quad (11)$$

Further, the equation of motion for V_α has the form

$$- \partial_\beta \left[\frac{1}{2v} e^{-2\Phi} (\mathbf{A}^{-1})_A^{\alpha\beta} \det \mathbf{A} \right] + J^\alpha = 0 , \quad (12)$$

where

$$J^\alpha = \int d^{p+1} \xi \frac{1}{v} \frac{\delta(\star \mathcal{G}_{(p+1)})}{\delta V_\alpha} \star \mathcal{G}_{(p+1)} . \quad (13)$$

Now we easily see that these equations of motion have the same form as the equation of motion derived from standard DBI and WZ action. In fact, inserting (6) into (10) and (13) we obtain

$$J_M^{WZ} = T \int d^{p+1}\xi \frac{\delta(\star\mathcal{G}_{(p+1)})}{\delta X^M}, \quad J_{WZ}^\alpha = T \int d^{p+1}\xi \frac{\delta(\star\mathcal{G}_{(p+1)})}{\delta V_\alpha} \quad (14)$$

that coincide with the currents introduced in [25]. Further, inserting (7) into (9) we obtain the equations of motion

$$\begin{aligned} & T \partial_M \Phi e^{-\Phi} \sqrt{-\det \mathbf{A}} - T e^{-\Phi} (\partial_M G_{KL} \partial_\alpha X^K \partial_\beta X^L + \partial_M B_{KL} \partial_\alpha X^K \partial_\beta X^L) (\mathbf{A}^{-1})^{\beta\alpha} \sqrt{-\det \mathbf{A}} \\ & + \partial_\alpha [T e^{-\Phi} g_{MN} \partial_\beta X^N (\mathbf{A}^{-1})_S^{\beta\alpha} \sqrt{-\det \mathbf{A}}] + \partial_\beta [T e^{-\Phi} b_{MN} \partial_\beta X^N (\mathbf{A}^{-1})_A^{\beta\alpha} \sqrt{-\det \mathbf{A}}] + J_M^{WZ} = 0, \end{aligned} \quad (15)$$

which are the equations of motion derived from the DBI and WZ action. In the same way we proceed with the equation of motion for V_α

$$T \partial_\beta [e^{-\Phi} (\mathbf{A}^{-1})_A^{\alpha\beta} \sqrt{-\det \mathbf{A}}] + J_{WZ}^\alpha = 0 \quad (16)$$

which is the equation of motion derived from DBI action. It is important to stress that given analysis is valid for $T \neq 0$. To see this explicitly note that (8) implies that $\det \mathbf{A} = 0$ for $T = 0$ and hence it is not possible to introduce inverse matrix to \mathbf{A} . In order to deal with this case it is convenient to proceed to the Hamiltonian formulation.

3 Hamiltonian Formalism for p -Brane With Dynamical Tension

In this section we perform the Hamiltonian formulation of p -brane with variable tension. For simplicity we consider the case of pure p -brane leaving the general analysis to the case of non-BPS Dp -brane that will be performed in the next section. In case of p -brane we have

$$\star \mathcal{G}_{(p+1)} = \frac{1}{p!} \epsilon^{\mu\mu_2 \dots \mu_{p+1}} \partial_\mu A_{\mu_2 \dots \mu_{p+1}} = \partial_\mu \omega^\mu, \quad (17)$$

where

$$\omega^\mu = \frac{1}{p!} \epsilon^{\mu\mu_1 \dots \mu_p} A_{\mu_1 \dots \mu_p} \quad (18)$$

is vector density of unit weight. With the help of ω^μ we can write the action for p -brane with variable tension in the form

$$S = \int d^{p+1}\xi \frac{1}{2v} (\det g + (\partial_\mu \omega^\mu)^2). \quad (19)$$

Now we proceed to the canonical formulation of this theory. From (19) we derive conjugate momenta

$$\begin{aligned} p_M &= \frac{\delta L}{\delta \partial_0 X^M} = \frac{1}{v} G_{MN} \partial_\mu X^N g^{\mu 0} \det g , \\ p_v &\approx 0 , \quad \tau_0 = \frac{\delta L}{\delta \partial_0 \omega^0} = \frac{1}{v} \partial_\mu \omega^\mu , \quad \tau_i = \frac{\delta L}{\delta \partial_0 \omega^i} \approx 0 . \end{aligned} \quad (20)$$

Then the bare Hamiltonian density is equal to

$$\begin{aligned} \mathcal{H}_B &= p_M \partial_0 X^M + \rho_0 \partial_0 \omega^0 - \mathcal{L} \\ &= \frac{v}{2 \det g_{ij}} p_M G^{MN} p_N + \frac{1}{2} \tau_0^2 v - \partial_i \omega^i \tau_0 \end{aligned} \quad (21)$$

using the fact that

$$p_M G^{MN} p_N = \frac{1}{v^2} \det g_{ij} \det g . \quad (22)$$

Further, from (20) we derive following primary constraints

$$\mathcal{H}_i = p_M \partial_i X^M \approx 0 . \quad (23)$$

As the result the Hamiltonian density with all primary constraints included has the form

$$\mathcal{H}_E = \frac{v}{2 \det g_{ij}} p_M G^{MN} p_N + \frac{1}{2} \tau_0^2 v - \partial_i \omega^i \tau_0 + N^i \mathcal{H}_i + U_v p_v + U^i \rho_i . \quad (24)$$

Now we have to study the stability of the primary constraints when the Hamiltonian that generates the time evolution is $H_E = \int d^p \xi \mathcal{H}_E$. The requirement of the preservation of the constraint $p_v \approx 0$ implies

$$\partial_t p_v = \{p_v, H_E\} = -\frac{1}{2 \det g_{ij}} \mathcal{H}_0 , \quad \mathcal{H}_0 = p_M G^{MN} p_N + \tau_0^2 \det g_{ij} \quad (25)$$

while the preservation of the constraint $\rho_i \approx 0$ implies an existence of additional constraints

$$\partial_0 \rho_i = \{\rho_i, H_E\} = \partial_i \tau^0 \equiv \mathcal{G}_i \approx 0 . \quad (26)$$

Finally we proceed to the analysis of the preservation of the constraints \mathcal{H}_i . We extend these constraints with the secondary constraints \mathcal{G}_i in order to ensure that they are preserved during the time evolution. In more details, let us introduce the constraints

$$\tilde{\mathcal{H}}_i = \mathcal{H}_i - \mathcal{G}_i \omega^0 \quad (27)$$

and its smeared form

$$\mathbf{H}_S(N^i) = \int d^p \xi N^i \tilde{\mathcal{H}}_i . \quad (28)$$

Now it is easy to see that this constraint has following non-zero Poisson brackets

$$\{\mathbf{H}_S(N^i), p_M\} = -\partial_i(N^i p_M), \quad \{\mathbf{H}_S(N^i), X^M\} = -N^i \partial_i X^M, \quad \{\mathbf{H}_S(N^i), \tau_0\} = -N^i \partial_i \tau_0 \quad (29)$$

and hence we find

$$\{\mathbf{H}_S(N^i), \mathcal{H}_0\} = -2\partial_i N^i \mathcal{H}_0 - N^i \partial_i \mathcal{H}_0, \quad \{\mathbf{H}_S(N^i), \mathbf{H}_S(M^j)\} = \mathbf{H}_S(N^j \partial_j M^i - \partial_j N^i M^j) \quad (30)$$

which implies that $\tilde{\mathcal{H}}_i$ are preserved during the time evolution of the system. Now in order to find the total Hamiltonian we have to include all constraints to it. We absorb the factor $\frac{v}{\det g_{ij}}$ into the Lagrange multiplier N_0 corresponding to the constraint \mathcal{H}_0 . In the same way we include ω^i into the definition of the Lagrange multiplier Γ^i corresponding to the constraint \mathcal{G}_i . As a result the total Hamiltonian has the form

$$H_T = \int d^p \xi (N_0 \mathcal{H}_0 + N^i \mathcal{H}_i + \Gamma^i \mathcal{G}_i), \quad (31)$$

where we do not induced the constraints $p_v \approx 0$, $\rho^i \approx 0$ since they decouple from the theory. Let us now consider the equation of motion for τ_0

$$\partial_0 \tau_0 = \{\tau_0, H_T\} = 0 \quad (32)$$

since H_T does not depend on ω^0 . Further, the constraint $\mathcal{G}_i \approx 0$ implies that $\partial_i \tau_0 = 0$ and consequently we find that $\tau_0 = T$ is a constant that can be identified with the tension of p-brane. Let us now determine remaining equations of motion

$$\begin{aligned} \partial_0 X^M &= \{X^M, H_T\} = 2N_0 G^{MN} p_N + N^i \partial_i X^M, \\ \partial_0 p_M &= \{p_M, H_T\} = -N_0 \partial_M G^{KL} p_K p_L \\ &\quad - N_0 \tau_0^2 \partial_M G_{KL} \partial_i X^K \partial_j X^L g^{ji} \det g_{ij} + 2\partial_i [N_0 \tau_0^2 G_{MN} \partial_j X^N g^{ji} \det g_{ij}] - \partial_i (N^i p_M). \end{aligned} \quad (33)$$

We see that for $\tau_0 = 0$ the equations of motion (33) simplify considerably

$$\begin{aligned} \partial_0 X^M &= \{X^M, H_T\} = 2N_0 G^{MN} p_N + N^i \partial_i X^M, \\ \partial_0 p_M &= \{p_M, H_T\} = -N_0 \partial_M G^{KL} p_K p_L - \partial_i (N^i p_M) \end{aligned} \quad (34)$$

together with the constraints $p_M \partial_i X^M \approx 0$, $p_M G^{MN} p_N \approx 0$. Clearly the diffeomorphism constraint can be solved by imposing $X^M = X^M(\xi^0)$. On the other hand we can consider more general dependence of

$$p_M = P_N(\xi^0) f(\xi^1, \dots, \xi^p), \quad N_0 = n(\xi^0) f^{-1}(\xi^1, \dots, \xi^p), \quad N^i = n^i(\xi^0) f^{-1}(\xi^1, \dots, \xi^p) \quad (35)$$

where $f(\xi^1, \dots, \xi^p)$ is arbitrary function [18]. Then it is easy to see that the equation of motion for X^M and P_M correspond to the equation of motion for tensionless particle where the localization of the particle along the world-volume of p-brane is determined by the function $f(\xi^1, \dots, \xi^p)$. On the other hand the physical meaning has the localization of this object in the target space time where the dynamics of the embedding modes is governed by the equations of motions for massless particle.

4 Non-BPS Dp-brane with Variable Tension

In this section we propose an action for non-BPS Dp-brane with variable tension. We claim that the Lagrangian density has the form

$$\mathcal{L}_{non} = \frac{1}{2v} \left[e^{-2\Phi} V^2(T) \det \tilde{\mathbf{A}} + (\star G_{(p+1)})^2 \right] , \quad (36)$$

where Φ is a dilaton, v is an independent worldvolume density and where

$$\tilde{\mathbf{A}}_{\mu\nu} = G_{MN} \partial_\mu X^M \partial_\nu X^N + \partial_\mu T \partial_\nu T + \mathcal{F}_{\mu\nu} , \quad (37)$$

where $\mathcal{F}_{\mu\nu} = B_{MN} \partial_\mu X^M \partial_\nu X^N + (\partial_\mu V_\nu - \partial_\nu V_\mu)$. Further, T is the tachyon field and $V(T)$ is a corresponding potential that is symmetric under $T \rightarrow -T$ and it has a maximum at $T = 0$ and has its minimum at $T = \pm\infty$ where it vanishes [13].

In case of the non-BPS Dp-brane we propose the scalar density $\star \mathcal{G}$ in the form

$$\star \tilde{\mathcal{G}} = \frac{1}{(p+1)!} \epsilon^{\mu_1 \dots \mu_{p+1}} \partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]} - \sum_{n \geq 0} \frac{1}{n! (2!)^n q!} \epsilon^{\mu_1 \dots \mu_{p+1}} V(T) (\mathcal{F})_{\mu_1 \dots \mu_{2n}}^n C_{\mu_{2n+1} \dots \mu_p} \partial_{\mu_{p+1}} T \quad (38)$$

where $q = p + 1 - 2n - 1$. Now we show that the equations of motion derived from the action $S_{non} = \int d^{p+1} \xi \mathcal{L}_{non}$ have the same form as the equations of motion derived from non-BPS Dp-brane action. First of all the equation of motion with respect to A implies

$$\frac{\star \tilde{\mathcal{G}}}{v} = \tau_p , \quad (39)$$

where τ_p can be interpreted as non-BPS Dp-brane tension. On the other hand the equation of motion with respect to v implies

$$e^{-2\Phi} V^2 \det \tilde{\mathbf{A}} + (\star \tilde{\mathcal{G}})^2 = 0 \quad (40)$$

that with the help of (39) implies (on condition that $\tau_p \neq 0$)

$$v = \frac{1}{\tau_p} e^{-\Phi} V(T) \sqrt{-\det \mathbf{A}} . \quad (41)$$

Finally we analyze the equations of motion for X^M and V_α and T . For the first one we obtain

$$\begin{aligned} & \frac{1}{2v} \partial_M [e^{-2\Phi}] V^2 \det \tilde{\mathbf{A}} + \frac{1}{2v} e^{-2\Phi} V^2 (\partial_M G_{KL} \partial_\alpha X^K \partial_\beta X^L + \partial_M B_{KL} \partial_\alpha X^K \partial_\beta X^L) (\tilde{\mathbf{A}}^{-1})^{\beta\alpha} \det \tilde{\mathbf{A}} \\ & - \partial_\alpha \left[\frac{1}{v} e^{-2\Phi} V^2 G_{MN} \partial_\beta X^N (\tilde{\mathbf{A}}^{-1})_S^{\beta\alpha} \det \tilde{\mathbf{A}} \right] - \partial_\beta \left[\frac{1}{v} e^{-2\Phi} V^2 B_{MN} \partial_\beta X^N (\tilde{\mathbf{A}}^{-1})_A^{\beta\alpha} \det \tilde{\mathbf{A}} \right] + \tilde{J}_M = 0 , \end{aligned} \quad (42)$$

where

$$\begin{aligned}
\tilde{J}_M &= \int d^{p+1} \xi \frac{1}{v} \frac{\delta(\star \tilde{\mathcal{G}})}{\delta X^M} \star \tilde{\mathcal{G}} = \\
&- \sum_{n \geq 0} \frac{1}{n!(2!)^n q!} \epsilon^{\mu_1 \dots \mu_{p+1}} \left(\frac{\star \tilde{\mathcal{G}}}{v} V(T) n \partial_M B_{KL} \partial_{\mu_1} X^K \partial_{\mu_2} X^L (\mathcal{F})_{\mu_3 \dots \mu_{2n}}^{n-1} C_{\mu_{2n+1} \dots \mu_p} \partial_{\mu_{p+1}} T \right. \\
&+ \frac{\star \tilde{\mathcal{G}}}{v} V(T) (\mathcal{F})_{\mu_1 \dots \mu_{2n}}^n \partial_M C_{K_1 \dots K_q} \partial_{\mu_{2n+1}} X^{K_1} \dots \partial_{\mu_q} X^{K_q} \partial_{\mu_{p+1}} T \\
&- 2n \partial_{\mu_1} \left[\frac{\star \tilde{\mathcal{G}}}{v} V(T) b_{MK} \partial_{\mu_2} X^K (\mathcal{F})_{\mu_3 \dots \mu_{2n}}^{n-1} C_{\mu_{2n+1} \dots \mu_p} \partial_{\mu_{p+1}} T \right] \\
&\left. - q \partial_{2n+1} \left[\frac{\star \tilde{\mathcal{G}}}{v} V(T) (\mathcal{F})_{\mu_1 \dots \mu_{2n}}^n C_{MK_2 \dots K_q} \partial_{\mu_{2n+2}} X^{M_2} \dots \partial_{\mu_p} X^{K_q} \partial_{\mu_{p+1}} T \right] \right) .
\end{aligned} \tag{43}$$

Further, the equations of motion for V_α have the form

$$- \partial_\beta \left[\frac{1}{v} e^{-2\Phi} V^2 (\tilde{\mathbf{A}}^{-1})_A^{\alpha\beta} \det \tilde{\mathbf{A}} \right] + \tilde{J}^\alpha = 0 , \tag{44}$$

where

$$\begin{aligned}
\tilde{J}^\alpha &= \int d^{p+1} \xi \frac{1}{v} \frac{\delta(\star \tilde{\mathcal{G}})}{\delta V_\alpha} \star \tilde{\mathcal{G}} \\
&= - \sum_{n \geq 0} \frac{2n}{n!(2!)^n q!} \partial_{\mu_2} \left[\frac{\star \mathcal{G}}{v} \epsilon^{\alpha \mu_2 \dots \mu_{p+1}} (\mathcal{F})_{\mu_3 \dots \mu_{2n}}^{n-1} C_{\mu_{2n+1} \dots \mu_p} \partial_{\mu_{p+1}} T \right] .
\end{aligned} \tag{45}$$

Finally the equation of motion for T has the form

$$e^{-2\Phi} \frac{dV}{dT} V \det \tilde{\mathbf{A}} - \partial_\alpha \left[\frac{1}{v} e^{-2\Phi} V^2 \partial_\beta T (\tilde{\mathbf{A}}^{-1})_S^{\beta\alpha} \det \tilde{\mathbf{A}} \right] + \tilde{J}_T = 0 , \tag{46}$$

where

$$\begin{aligned}
\tilde{J}_T &= \int d^{p+1} \xi \frac{1}{v} \frac{\delta(\star \tilde{\mathcal{G}})}{\delta T} \star \tilde{\mathcal{G}} = \\
&= \sum_{n \geq 0} \frac{1}{n!(2!)^n q!} \epsilon^{\mu_1 \dots \mu_{p+1}} \left(\frac{\star \tilde{\mathcal{G}}}{v} V'(T) (\mathcal{F})_{\mu_1 \dots \mu_{2n}}^n C_{\mu_{2n+1} \dots \mu_p} \partial_{\mu_{p+1}} T \right. \\
&\left. - \partial_{\mu_{p+1}} \left[\frac{\star \mathcal{G}}{v} V(T) (\mathcal{F})_{\mu_1 \dots \mu_{2n}}^n C_{\mu_{2n+1} \dots \mu_p} \right] \right)
\end{aligned} \tag{47}$$

Now we show that the equations of motion (39), (40), (42) and (46) can be solved with tachyon kink ansatz that can be interpreted as a lower dimensional D(p-1)-brane,

following the similar analysis presented in case of non-BPS Dp-brane in [14, 15]. We choose one world-volume coordinate, say $\xi^p \equiv x$ and consider following ansatz for the tachyon

$$T(x, \xi^{\hat{\mu}}) = f(a(x - t(\xi^{\hat{\mu}}))) , \quad (48)$$

where as in [14] we presume that $f(u)$ satisfies following properties

$$f(-u) = -f(u) , \quad f'(u) > 0 , \forall u , \quad f(\pm\infty) = \pm\infty , \quad (49)$$

which is however an arbitrary function of its argument u . a is a constant which can be taken to ∞ in the end that leads to the configuration when $T = \infty$ for $x > t(\xi^{\hat{\mu}})$ and $T = -\infty$ for $x < t(\xi^{\hat{\mu}})$. Finally note that $\hat{\mu} = 0, \dots, p-1$. Let us also presume following ansatz for massless fields

$$X^M(x, \xi^{\hat{\mu}}) = X^M(\xi^{\hat{\mu}}) , \quad A_x(x, \xi^{\hat{\mu}}) = 0 , \quad A_{\hat{\mu}}(x, \xi^{\hat{\mu}}) = A_{\hat{\mu}}(\xi^{\hat{\mu}}) . \quad (50)$$

Our goal is to show that the dynamics of the kink is governed by D(p-1)-brane equations of motion derived in section (2). As the first step we determine the matrix $\tilde{\mathbf{A}}$ for the ansatz (48) and (50)

$$\tilde{\mathbf{A}}_{\mu\nu} = \begin{pmatrix} \mathbf{A}_{\hat{\mu}\hat{\nu}} + a^2 f'^2 \partial_{\hat{\mu}} t \partial_{\hat{\nu}} t & -a^2 f'^2 \partial_{\hat{\nu}} t \\ -a^2 f'^2 \partial_{\hat{\mu}} t & a^2 f'^2 \end{pmatrix} \quad (51)$$

where

$$\mathbf{A}_{\hat{\mu}\hat{\nu}} = G_{MN} \partial_{\hat{\mu}} X^M \partial_{\hat{\nu}} X^N + \mathcal{F}_{\hat{\mu}\hat{\nu}} , \quad \mathcal{F}_{\hat{\mu}\hat{\nu}} = B_{MN} \partial_{\hat{\mu}} X^M \partial_{\hat{\nu}} X^N + (\partial_{\hat{\mu}} V_{\hat{\nu}} - \partial_{\hat{\nu}} V_{\hat{\mu}}) . \quad (52)$$

Note that for the matrix (52) the determinant $\det \tilde{\mathbf{A}}$ has a form

$$\det \tilde{\mathbf{A}} = \det(\tilde{\mathbf{A}}_{\hat{\mu}\hat{\nu}} - \tilde{\mathbf{A}}_{\hat{\mu}x} \frac{1}{\tilde{\mathbf{A}}_{xx}} \tilde{\mathbf{A}}_{x\hat{\nu}}) \tilde{\mathbf{A}}_{xx} = a^2 f'^2 \det \mathbf{A}_{\hat{\mu}\hat{\nu}} . \quad (53)$$

As the next step we determine inverse matrix $\tilde{\mathbf{A}}^{-1}$ that has following exact form (for all a)

$$\begin{aligned} (\tilde{\mathbf{A}}^{-1})^{\hat{\mu}\hat{\nu}} &= (\mathbf{A}^{-1})^{\hat{\mu}\hat{\nu}} , \quad (\tilde{\mathbf{A}}^{-1})^{xx} = \frac{1 + a^2 f'^2 \partial_{\hat{\mu}} t (\mathbf{A}^{-1})^{\hat{\mu}\hat{\nu}} \partial_{\hat{\nu}} t}{a^2 f'^2} , \\ (\tilde{\mathbf{A}}^{-1})^{\hat{\mu}x} &= (\mathbf{A}^{-1})^{\hat{\mu}\hat{\nu}} \partial_{\hat{\nu}} t , \quad (\tilde{\mathbf{A}}^{-1})^{x\hat{\nu}} = \partial_{\hat{\mu}} t (\mathbf{A}^{-1})^{\hat{\mu}\hat{\nu}} . \end{aligned} \quad (54)$$

Now we show that the ansatz (48) and (50) is solution of the equation of motion. First of all using (54) we easily find

$$\partial_{\mu} T(\tilde{\mathbf{A}}^{-1})^{\mu\hat{\nu}}_S = 0 , \quad \partial_{\mu} T(\tilde{\mathbf{A}}^{-1})^{\mu x}_S = \frac{1}{a f'} . \quad (55)$$

Then it is easy to see that the equation of motion for the tachyon is satisfied since

$$\frac{1}{v}e^{-2\Phi}\frac{dV}{dT}V\det\tilde{\mathbf{A}} - \partial_\alpha\left[\frac{1}{v}e^{-2\Phi}V^2\partial_\beta T(\tilde{\mathbf{A}}^{-1})^\beta_\alpha\det\tilde{\mathbf{A}}\right] + \tilde{J}_T = 0, \quad (56)$$

where we also used (41) together with the fact that the current \tilde{J}_T vanishes identically for the ansatz (48) and (50). Now we can proceed to the equations of motion for X^M . Using the fact that X^M depend on $\xi^{\hat{\mu}}$ only and the explicit form of the inverse matrix $\tilde{\mathbf{A}}^{-1}$ given in (54) we obtain that the equation (42) has the form

$$\begin{aligned} & - \tau_p V a f' \partial_M [e^{-\Phi}] \sqrt{-\det \mathbf{A}_{\hat{\mu}\hat{\nu}}} \\ & - a f' \tau_p V \frac{1}{2} e^{-\Phi} (\partial_M G_{KL} \partial_{\hat{\mu}} X^K \partial_{\hat{\nu}} X^L + \partial_M B_{KL} \partial_{\hat{\mu}} X^K \partial_{\hat{\nu}} X^L) (\mathbf{A}^{-1})^{\hat{\nu}\hat{\mu}} \sqrt{-\det \mathbf{A}_{\hat{\mu}\hat{\nu}}} \\ & + a f' V \tau_p \partial_{\hat{\nu}} [e^{-\Phi} G_{MN} \partial_{\hat{\mu}} X^N (\tilde{\mathbf{A}}^{-1})^{\hat{\mu}\hat{\nu}}_S \sqrt{-\det \mathbf{A}_{\hat{\mu}\hat{\nu}}}] + a f' V \partial_{\hat{\nu}} [e^{-\Phi} B_{MN} \partial_{\hat{\mu}} X^N (\tilde{\mathbf{A}}^{-1})^{\hat{\mu}\hat{\nu}}_A \sqrt{-\det \mathbf{A}_{\hat{\mu}\hat{\nu}}}] \\ & + \tilde{J}_M = 0, \end{aligned} \quad (57)$$

where we also used the fact that for any function $F(a(x - t(\xi^{\hat{\mu}})))$ we have

$$\partial_{\hat{\mu}} F(a(x - t(\xi^{\hat{\mu}}))) = -\partial_x F(a(x - t(\xi^{\hat{\mu}}))) \partial_{\hat{\mu}} t. \quad (58)$$

Let us now calculate \tilde{J}_M . It is easy to see that the only non-zero contribution contains x -derivative of T since in the opposite case the presence of the totally antisymmetric tensor ϵ implies x -derivative of massless fields which are zero by definition. Then \tilde{J}^M is equal to

$$\tilde{J}^M = V a f' J_{WZ}^M, \quad (59)$$

where J_{WZ}^M is the current for D(p-1)-brane (14). Collecting all these results together we obtain that the equations of motion for X^M have the form

$$\begin{aligned} & V a f' \left(-\tau_p \partial_M [e^{-\Phi}] \sqrt{-\det \mathbf{A}_{\hat{\mu}\hat{\nu}}} \right. \\ & - \tau_p \frac{1}{2} e^{-\Phi} (\partial_M G_{KL} \partial_{\hat{\mu}} X^K \partial_{\hat{\nu}} X^L + \partial_M B_{KL} \partial_{\hat{\mu}} X^K \partial_{\hat{\nu}} X^L) (\mathbf{A}^{-1})^{\hat{\nu}\hat{\mu}} \sqrt{-\det \mathbf{A}_{\hat{\mu}\hat{\nu}}} \\ & + \left. \tau_p \partial_{\hat{\nu}} [e^{-\Phi} G_{MN} \partial_{\hat{\mu}} X^N (\tilde{\mathbf{A}}^{-1})^{\hat{\mu}\hat{\nu}}_S \sqrt{-\det \mathbf{A}_{\hat{\mu}\hat{\nu}}}] + \partial_{\hat{\nu}} [e^{-\Phi} B_{MN} \partial_{\hat{\mu}} X^N (\tilde{\mathbf{A}}^{-1})^{\hat{\mu}\hat{\nu}}_A \sqrt{-\det \mathbf{A}_{\hat{\mu}\hat{\nu}}}] + J_M^{WZ} \right) = 0 \end{aligned} \quad (60)$$

with following physical interpretation. Since for large T the potential $V(T)$ goes as $V \sim e^{-T^2}$ we easily see that aV is zero for all points $x \neq t(\xi^{\hat{\mu}})$ in the limit $a \rightarrow \infty$. In other words the kink is localized at the point $x = t(\xi^{\hat{\mu}})$. It is important to stress that $t(\xi^{\hat{\mu}})$ does not have any physical meaning since we consider manifestly world-sheet diffeomorphism invariant theory and hence this kink can be localized at any point on the world-volume of non-BPS Dp-brane. On the other hand we see that at the point $x = t(\xi^{\hat{\mu}})$ the equation of motion for X^M are satisfied on condition when

the expression in the bracket (...) is zero. However this expression is exactly the equations of motion (15) for D(p-1)-brane.

In the same way we proceed with the equation of motion for V_x and $V_{\hat{\mu}}$. In the first case we find that the equation of motion (44) has the form

$$af'V\partial_{\hat{\nu}}t\left(\partial_{\hat{\mu}}[e^{-\Phi}(\mathbf{A}^{-1})^{\hat{\mu}\hat{\nu}}\sqrt{-\det\mathbf{A}_{\hat{\mu}\hat{\nu}}}] + J_{WZ}^{\hat{\nu}}\right) = 0 , \quad (61)$$

where we used the fact that \tilde{J}^x evaluated on the ansatz (48),(50) is equal to

$$\tilde{J}^x = -af'V\sum_{n\geq 0}\frac{2n}{(n)!(2!)^nq!}\epsilon^{\hat{\nu}\hat{\mu}_2\ldots\hat{\mu}_p x}\partial_{\hat{\mu}_2}[(\mathcal{F})_{\hat{\mu}_3\ldots\hat{\mu}_{2n}}^{n-1}C_{\hat{\mu}_{2n+1}\ldots\hat{\mu}_p}]\partial_{\hat{\nu}}t = af'VJ_{WZ}^{\hat{\nu}}\partial_{\hat{\nu}}t . \quad (62)$$

In the similar way we can proceed with $V_{\hat{\mu}}$ and we find that the equation (44) has the form

$$\tau_p V a f' \left(\partial_{\hat{\nu}} [e^{-\Phi}(\mathbf{A}^{-1})_A^{\hat{\mu}\hat{\nu}} \sqrt{-\det\mathbf{A}_{\hat{\mu}\hat{\nu}}}] + J_{WZ}^{\hat{\mu}} \right) = 0 . \quad (63)$$

The physical interpretation of the equations (61) and (63) is the same as in case of the equation of motion for X^M . Explicitly, these equations are valid at all points $x \neq t(\xi^{\hat{\mu}})$ while at $x = t(\xi^{\hat{\mu}})$ they are obeyed on condition that the expression in the bracket is zero. However this is exactly the equation of motion (16) for the gauge field on the world-volume of D(p-1)-brane. In summary we have found that the tachyon kink on the world-volume of non-BPS Dp-brane with dynamical tension corresponds to the lower dimensional stable D(p-1)-brane. It is important to stress that given analysis is valid on condition that $\tau_p \neq 0$. In order to analyze solution with $\tau_p = 0$ we have to switch to the Hamiltonian formulation of non-BPS Dp-brane with dynamical tension.

5 Hamiltonian Formalism for non-BPS Dp-brane with Dynamical Tension

In order to find solution with $\tau_p = 0$ it is useful to pass to the Hamiltonian formalism. For simplicity we presume zero RR background so that the action has the form

$$S = \int d^{p+1}\xi \frac{1}{2v} (e^{-2\Phi} V^2 \det \tilde{\mathbf{A}} + (\partial_{\mu}\omega^{\mu})^2) . \quad (64)$$

From this action we find conjugate momenta

$$\begin{aligned}
p_M &= \frac{1}{v} V^2 e^{-2\Phi} (G_{MN} \partial_\mu X^M (\tilde{\mathbf{A}}^{-1})_S^{\mu 0} + B_{MN} \partial_\mu X^N (\tilde{\mathbf{A}}^{-1})_A^{\mu 0}) \det \tilde{\mathbf{A}} , \\
\pi^i &= \frac{1}{v} V^2 e^{-2\Phi} (\tilde{\mathbf{A}}^{-1})_A^{i0} \det \tilde{\mathbf{A}} , \quad \pi^0 \approx 0 , \\
\tau_0 &= \frac{\delta L}{\delta \partial_0 \omega^0} = \frac{1}{v} \partial_\mu \omega^\mu , \quad \tau_i = \frac{\delta L}{\delta \partial_0 \omega^i} \approx 0 . \\
p_T &= \frac{\delta L}{\delta \partial_0 T} = \frac{1}{v} e^{-2\Phi} V^2 \partial_\beta T (\tilde{\mathbf{A}}^{-1})_S^{\beta 0} \det \tilde{\mathbf{A}} .
\end{aligned} \tag{65}$$

Using these relations we easily find that the bare Hamiltonian is equal to

$$H_B = \int d^p \xi \left(\frac{e^{-2\Phi}}{2v} V^2 \det \tilde{\mathbf{A}} + \frac{\tau_0^2}{2} v - \partial_i \omega^i \tau_0 + \partial_i V_0 \pi^i \right) . \tag{66}$$

In order to express $\det \tilde{\mathbf{A}}$ as a function of the canonical variables we use the fact that

$$\Pi_M G^{MN} \Pi_N + p_T^2 + \pi^i (\tilde{\mathbf{A}}_S)_{ij} \pi^j = \frac{e^{-4\Phi} V^2}{v^2} \det \tilde{\mathbf{A}}_{ij} \det \tilde{\mathbf{A}} , \tag{67}$$

where

$$\Pi_M \equiv p_M - B_{MN} \partial_i X^N \pi^i . \tag{68}$$

Note also that (65) imply following primary constraint

$$\mathcal{H}_i \equiv p_M \partial_i X^M + F_{ij} \pi^j + p_T \partial_i T \approx 0 . \tag{69}$$

Now we are ready to write an extended form of the Hamiltonian with all primary constraints included

$$H_E = \int d^p \xi \left(\frac{v e^{2\Phi}}{2 \det \tilde{\mathbf{A}}_{ij} V^2} \tilde{\mathcal{H}}_0 + \omega^i \partial_i \tau_0 + N^i \mathcal{H}_i - V_0 \partial_i \pi^i \right) , \tag{70}$$

where

$$\tilde{\mathcal{H}}_0 = \Pi_M G^{MN} \Pi_N + p_T^2 + \pi^i (\tilde{\mathbf{A}}_S)_{ij} \pi^j + V^2 \tau_0^2 e^{-2\Phi} \det \tilde{\mathbf{A}}_{ij} . \tag{71}$$

Now the requirement of the preservation of the primary constraints $p_v \approx 0$, $\rho_i \approx 0$ and π^0 implies following secondary constraints

$$\mathcal{H}_0 \approx 0 , \mathcal{G}_i = \partial_i \tau_0 \approx 0 , \mathcal{G}_V = \partial_i \pi^i \approx 0 \tag{72}$$

so that the total Hamiltonian with all constraints included has the form

$$H_T = \int d^p \xi (N_0 \mathcal{H}_0 + N^i \tilde{\mathcal{H}}_i + V^i \mathcal{G}_i + U^v \mathcal{G}_v + \rho^v p_v + \rho^i \tau_i) , \quad (73)$$

where the extended diffeomorphism constraint has the form

$$\tilde{\mathcal{H}}_i = p_N \partial_i X^M + F_{ij} \pi^j + p_T \partial_i T - \partial_i \tau_0 \omega^0 . \quad (74)$$

Finally we have to show that all constraints are preserved during the time evolution of the system. To do this we introduce smeared form of these constraints

$$\begin{aligned} \mathbf{H}_T(N) &= \int d^p \xi N \tilde{\mathcal{H}}_0 , & \mathbf{H}_S(N^i) &= \int d^p \xi N^i \tilde{\mathcal{H}}_i , \\ \mathbf{G}_A(V) &= \int d^p \xi V \mathcal{G}_V , & \mathbf{G}(W^i) &= \int d^p \xi W^i \mathcal{G}_i . \end{aligned} \quad (75)$$

Now with the help of the following Poisson brackets

$$\begin{aligned} \{\mathbf{H}_S(N^i), X^M\} &= -N^i \partial_i X^M , & \{\mathbf{H}_S(N^i), p_M\} &= -\partial_i (N^i p_M) , \\ \{\mathbf{H}_S(N^i), A_i\} &= -N^k F_{ki} , & \{\mathbf{H}_S(N^i), \pi^i\} &= -\partial_k (N^k \pi^i) + \partial_l N^i \pi^l + N^i \mathcal{G}_V , \\ \{\mathbf{H}_S(N^i), T\} &= -N^i \partial_i T , & \{\mathbf{H}_S(N^i), p_T\} &= -\partial_i (N^i p_T) \end{aligned} \quad (76)$$

we obtain

$$\{\mathbf{H}_S(N^i), \mathbf{H}_T(M)\} = \mathbf{H}_T(N^i \partial_i M) + 2\mathbf{G}_A(M N^i (\tilde{\mathbf{A}}_S)_{ij} \pi^j + M \Pi^M B_{MN} \partial_i X^N) , \quad (77)$$

and also

$$\{\mathbf{H}_S(N^i), \mathbf{H}_S(M^j)\} = \mathbf{H}_S(N^j \partial_j M^i - \partial_j N^i M^j) + \mathbf{G}_A(M^i F_{ij} N^j) . \quad (78)$$

Finally we calculate the Poisson bracket $\{\mathbf{H}_T(N), \mathbf{H}_T(M)\}$. Using the fact that

$$\{\Pi_M(\xi), \Pi_N(\xi')\} = H_{MNK} \partial_i X^K \delta(\xi - \xi') + B_{MN} \mathcal{G} \delta(\xi - \xi') \quad (79)$$

and after some tedious calculations we obtain the result

$$\begin{aligned} \{\mathbf{H}_T(N), \mathbf{H}_T(M)\} &= \mathbf{H}_S((N \partial_j M - M \partial_j N) \hat{\mathbf{A}}_S^{ji} V^2 \tau_0^2 e^{-2\Phi} \det \tilde{\mathbf{A}}_{ij}) + \\ &+ \mathbf{G}((N \partial_j M - M \partial_j N) \omega^0 \hat{\mathbf{A}}_S^{ji} V^2 \tau_0^2 e^{-2\Phi} \det \tilde{\mathbf{A}}_{ij}) + \\ &+ 4\mathbf{H}_S((N \partial_j M - M \partial_j N) \pi^j \pi^i) + 4\mathbf{G}_A((N \partial_j M - M \partial_j N) \pi^i \pi^j \omega^0) . \end{aligned} \quad (80)$$

From the previous Poisson brackets we see that $\mathcal{H}_0, \tilde{\mathcal{H}}_i, \mathcal{G}, \mathcal{G}_i$ are the first class constraints and no new constraints are generated during the time evolution of the

system. Now we proceed to the solution of the equations of motion for τ_0, A_i and π^i that follow from the Hamiltonian (73)

$$\begin{aligned}
\partial_0 \tau_0 &= \{\tau_0, H_T\} = 0, \\
\partial_0 A_i &= \{A_i, H_T\} = -2N_0 B_{MN} \partial_i X^N G^{MK} \Pi_K + (\tilde{\mathbf{A}}_S)_{ij} \pi^j + \partial_i V_0 + N^j F_{ji}, \\
\partial_0 \pi^i &= \{\pi^i, H_T\} = -2\partial_k [V^2 \tau_0^2 e^{-2\Phi} \hat{\mathbf{A}}_A^{ki} \det \tilde{\mathbf{A}}_{ij}] + \partial_k [N^k \pi^i - N^i \pi^k],
\end{aligned} \tag{81}$$

where $\hat{\mathbf{A}}^{ki}$ is the matrix inverse to $\tilde{\mathbf{A}}_{ij} \hat{\mathbf{A}}^{jk} = \delta_i^k$. Further, the equations of motion for T and p_T have the form

$$\begin{aligned}
\partial_0 T &= \{T, H_T\} = 2N_0 p_T + N^i \partial_i T, \\
\partial_0 p_T &= \{p_T, H_T\} \\
&= 2\partial_i (\pi^i \partial_j T \pi^j) - 2V \frac{dV}{dT} \tau_0^2 e^{-2\Phi} \det \tilde{\mathbf{A}}_{ij} + \partial_i [V^2 e^{-2\Phi} \partial_j T \hat{\mathbf{A}}^{ji} \det \tilde{\mathbf{A}}] + \partial_i [N^i p_T].
\end{aligned} \tag{82}$$

Finally we determine equations of motion for X^M and p_M

$$\begin{aligned}
\partial_0 X^M &= \{X^M, H_T\} = 2N_0 G^{MN} \Pi_N + N^i \partial_i X^M, \\
\partial_0 p_M &= \{p_M, H_T\} = N_0 \Pi_K \partial_M G^{KL} \Pi_L + 2N_0 \partial_M B_{KL} \partial_i X^L \pi^i G^{KN} \Pi_N - 2\partial_i [N_0 B_{KM} \pi^i G^{KN} \Pi_N] \\
&\quad - N_0 \pi^i \partial_i X^K \partial_M G_{KL} \partial_j X^L \pi^j + 2\partial_i [N_0 \pi^i G_{MN} \partial_j X^N \pi^j] - N_0 V^2 \tau_0^2 \partial_M [e^{-2\Phi}] \det \tilde{\mathbf{A}}_{ij} - \\
&\quad - N_0 V^2 e^{-2\Phi} \tau_0^2 (\partial_i X^K \partial_M G_{KL} \partial_j X^L + \partial_i X^K \partial_M B_{KL} \partial_j X^L) \hat{\mathbf{A}}^{ji} \det \tilde{\mathbf{A}}_{ij} \\
&\quad + 2\partial_i [N_0 V^2 e^{-2\Phi} \tau_0^2 G_{MN} \partial_j X^N \hat{\mathbf{A}}_S^{ji} \det \tilde{\mathbf{A}}_{ij}] + 2\partial_i [N_0 V^2 e^{-2\Phi} \tau_0^2 B_{MN} \hat{\mathbf{A}}_A^{ji} \det \tilde{\mathbf{A}}_{ij}] + \partial_i [N^i p_M].
\end{aligned} \tag{83}$$

We will analyze these equations of motion for two possible configurations. The first one corresponding to the tachyon vacuum and the second one corresponding to the zero tension limit.

5.1 Tachyon Vacuum Solution

It is easy to see that $T = T_{min}, p_T = 0$ where $\frac{dV}{dT}(T_{min}) = 0, V(T_{min}) = 0$ is the solutions of the equation of motion. Further, the equation of motion (81) together with the constraints \mathcal{G}_i implies that τ_0 is constant. In this case the remaining equations of motion simplify considerably

$$\begin{aligned}
\partial_0 A_i &= -2N_0 B_{MN} \partial_i X^N G^{MK} \Pi_K + 2N_0 (\tilde{\mathbf{A}}_S)_{ij} \pi^j + \partial_i V_0 + N^j F_{ji}, \\
\partial_0 \pi^i &= \partial_k [N^k \pi^i - N^i \pi^k], \\
\partial_0 X^M &= \{X^M, H_T\} = 2N_0 G^{MN} \Pi_N + N^i \partial_i X^M, \\
\partial_0 p_M &= N_0 \Pi_K \partial_M G_{KL} \Pi_L + 2N_0 \partial_M B_{KL} \partial_i X^L \pi^i G^{KN} \Pi_N - 2\partial_i [N_0 B_{KM} \pi^i G^{KN} \Pi_N] - \\
&\quad - N_0 \pi^i \partial_i X^K \partial_M G_{KL} \partial_j X^L \pi^j + 2\partial_i [N_0 \pi^i G_{MN} \partial_j X^N \pi^j] + \partial_i [N^i p_M].
\end{aligned} \tag{84}$$

To proceed further we introduce following projector

$$\Delta_j^i = \delta_j^i - \frac{g_{jk}\pi^k\pi^i}{\pi^m g_{mn}\pi^n} , \quad g_{mn} = G_{MN}\partial_m X^M \partial_n X^N \quad (85)$$

that obeys the relation $\Delta_j^i \pi^j = 0$ and $\Delta_k^i \Delta_j^k = \Delta_j^i$. In other words it is a projector on directions transverse to π^i . Then we can write

$$N^i = \Delta_j^i N^j + \frac{N^j g_{jk}\pi^k\pi^i}{\pi^m g_{mn}\pi^n} \equiv N_\perp^i + N_{II}\pi^i , \quad (86)$$

where by definition $N_\perp^i g_{ij}\pi^j = 0$. Before we proceed further we should mention that π^i has the physical dimension $[\pi^i] = L^{-(p+1)}$ where L is some length scale. Then is convenient to introduce dimensionless $\tilde{\pi}^i$ when we write $\pi^i = \tilde{\pi}^i \tau_p$. Using this notation we can introduce following derivative

$$\pi^i \partial_i = \tau_p \partial_\sigma . \quad (87)$$

Now we return to the equation of motion for π^i where we use the split (86)

$$\partial_0 \pi^i = \partial_k N_\perp^k \pi^i + \tau_p \partial_\sigma N_{II} \pi^i - \tau_p \partial_\sigma N_\perp^i - \tau_p \partial_\sigma N_{II} \pi^i - \tau_p \partial_\sigma \pi^i N_{II} . \quad (88)$$

Our goal is to find solution of this equation when π^i are constants. In this case the constraint \mathcal{G}_V is automatically obeyed while the equation above takes the form

$$0 = \partial_k N_\perp^k \pi^i - \tau_p \partial_\sigma N_\perp^i \quad (89)$$

that is obeyed for all i on condition when $N_\perp^i = \text{const}$ that without lost of generality can be taken to be equal to zero. This choice also simplifies considerably the equation of motion for X^M, p_M ⁵

$$\begin{aligned} \partial_0 X^M &= 2N_0 G^{MN} \Pi_N + \tau_p N_{II} \partial_\sigma X^M , \\ \partial_0 p_M &= N_0 \Pi_K \partial_M G^{KL} \Pi_L + 2\tau_p N_0 \partial_M B_{KL} \partial_\sigma X^L G^{KN} \Pi_N - 2\tau_p \partial_\sigma [N_0 B_{KM} G^{KN} \Pi_N] \\ &\quad - \tau_p N_0 \partial_\sigma X^K \partial_M G_{KL} \partial_\sigma X^L + 2\tau_p \partial_\sigma [N_0 G_{MN} \partial_\sigma X^N] + \partial_\sigma [N_{II} p_M] . \end{aligned} \quad (90)$$

Now we argue that given system of the equations of motion possesses fundamental string solution. Note that the Nambu-Gotto action for the fundamental string in general background has the form

$$S = -\tau_F \int d\tau d\sigma [\sqrt{-\det \gamma_{\alpha\beta}} - B_{MN} \partial_\tau X^M \partial_\sigma X^N] , \quad (91)$$

⁵Note that the equations of motion for A_i determine the time evolution of A_i as functions of p_M and X^M . For that reason we will not analyze it explicitly.

where

$$\gamma_{\alpha\beta} = G_{MN}\partial_\alpha Z^M \partial_\beta Z^N, \alpha, \beta = \tau, \sigma. \quad (92)$$

It is easy to find the Hamiltonian from (91). Explicitly, from (91) we find momenta conjugate to Z^M as

$$K_M = \frac{\delta \mathcal{L}_{NG}}{\delta \partial_\tau Z^M} = \tau_F \frac{G_{MN} \partial_\tau Z^N \gamma_{\sigma\sigma} - G_{MN} \partial_\sigma Z^N \gamma_{\tau\sigma}}{\sqrt{-\det \gamma}} + \tau_F B_{MN} \partial_\sigma Z^N. \quad (93)$$

Then the bare Hamiltonian \mathbf{K} is zero

$$\mathbf{K} = \int d\sigma (K_M \partial_\tau Z^M - \mathcal{L}_{NG}) = 0 \quad (94)$$

while using (93) we find two primary constraints

$$\mathcal{K}_\sigma \equiv K_M \partial_\sigma Z^M \approx 0, \quad \mathcal{K}_\tau \equiv \frac{1}{\tau_F} \Psi_M G^{MN} \Psi_N + \tau_F \gamma_{\sigma\sigma} \approx 0, \quad (95)$$

where we defined

$$\Psi_M = (K_M - \tau_F B_{MK} \partial_\sigma Z^K). \quad (96)$$

Then the extended Hamiltonian has the form

$$\mathbf{K}_E = \int d\sigma (\lambda_\tau \mathcal{K}_\tau + \lambda_\sigma \mathcal{K}_\sigma), \quad (97)$$

where $\lambda_\tau, \lambda_\sigma$ are dimensionless Lagrange multipliers since $\mathcal{K}_\tau, \mathcal{K}_\sigma$ have the physical dimensions L^{-2} . Using (97) we derive following equations of motions for Z^M, K_M

$$\begin{aligned} \partial_\tau Z^M &= \{Z^M, \mathbf{K}_E\} = \frac{2}{\tau_F} \lambda_\tau G^{MN} \Psi_N + \lambda_\sigma \partial_\sigma Z^M, \\ \partial_\tau K_M &= \{K_M, \mathbf{K}_E\} = \frac{\lambda_\tau}{\tau_F} \Psi_P \partial_M G^{PN} \Psi_N \\ &\quad + 2\lambda_\tau \partial_M B_{NK} \partial_\sigma Z^K G^{NP} \Psi_P - 2\partial_\sigma [\lambda_\tau B_{NM} G^{NP} \Psi_P] - \\ &\quad - \lambda_\tau \tau_F \partial_M G_{KL} \partial_\sigma Z^K \partial_\sigma Z^L + 2\tau_F \partial_\sigma [\lambda_\tau G_{MN} \partial_\sigma Z^L]. \end{aligned} \quad (98)$$

Now we see that the non-BPS Dp-brane at the tachyon vacuum with constant electric flux possesses fundamental string solution on condition when we identify Z^M with X^M and $p_M = \frac{\tau_p}{\tau_F} K_M$ together with $N_{II} = \frac{1}{\tau_p} \lambda_\sigma, N_0 = \frac{1}{\tau_p} \lambda_\tau$. It is very interesting that this solution does not depend on all world-volume coordinates of non-BPS Dp-brane but it only depends on σ , where σ is defined by the orientation of the electric flux on the world-volume of non-BPS Dp-brane at the tachyon vacuum. We mean that this is a natural result if we recognize that it is believed that at the tachyon vacuum the non-BPS Dp-brane disappears. Further, note that the localization of the electric flux on the world-volume of non-BPS Dp-brane does not have physical meaning when the full world-volume diffeomorphism invariance is preserved.

5.2 Zero Tension Solution

Now we will discuss another interesting special solution corresponding to the case when $\tau_0 = 0$. Then $\tilde{\mathcal{H}}_0$ has simplifies considerably and has the form

$$\tilde{\mathcal{H}}_0 = \Pi_M G^{MN} \Pi_N + p_T^2 + \pi^i (\tilde{\mathbf{A}}_S)_{ij} \pi^j . \quad (99)$$

Let us now combine T with X^M into $\mathbf{Z}^A = (X^M, T)$ so that

$$\tilde{\mathcal{H}}_0 = \Pi_A \mathbf{G}^{AB} \Pi_B + \pi^i \gamma_{ij} \pi^j , \quad \tilde{\mathcal{H}}_i = \mathbf{p}_A \partial_i \mathbf{Z}^A , \quad (100)$$

where

$$\Pi_M = \Pi_M , \Pi_T = p_T , \gamma_{ij} = \partial_i \mathbf{Z}^A \mathbf{G}_{AB} \partial_j \mathbf{Z}^B , \mathbf{G}^{MN} = G^{MN} , \mathbf{G}^{TT} = 1 . \quad (101)$$

Now we see that the Hamiltonian density for the tensionless Dp-brane has almost the same form as the Hamiltonian density of the non-BPS Dp-brane at the tachyon vacuum with exception that there is an additional embedding mode T . In other words tensionless non-BPS Dp-brane propagates in the space-time with an additional dimensions. It is also clear that this theory possesses fundamental string solution where again this string propagates in the higher dimensional space-time. Finally we should also stress one important point. Naively we could expect that the tachyon condensation on tensionless non-BPS Dp-brane in the form of the kink solution could lead to an emergence of tensionless D(p-1)-brane. However as follows from the form of the Hamiltonian constraint (100) we see that there is no tachyon potential in the tensionless limit and hence it is not possible to find the tachyon kink solution with the interpretation as a lower dimensional tensionless D(p-1)-brane. This has also nice physical interpretation since zero tension solution is very similar to the tachyon vacuum solution where we argued unstable Dp-brane should disappear and hence it does not make sense to speak about lower dimensional tensionless D(p-1)-brane.

5.3 Inclusion of RR fields: The Case of non-BPS D2-brane

In this section we perform Hamiltonian analysis of non-BPS D2-brane with the presence of the background Ramond-Ramond fields. We consider this specific example in order to simplify the form of $\star \tilde{\mathcal{G}}$ keeping in mind that the generalization of this analysis to the more general case is straightforward. Explicitly, in case of non-zero RR fields $\star \tilde{\mathcal{G}}$ for unstable D2-brane has the form

$$\star \tilde{\mathcal{G}} = \partial_\mu \omega^\mu - \epsilon^{\mu_1 \mu_2 \mu_3} \left(\frac{1}{2} V(T) C_{\mu_1 \mu_2} \partial_{\mu_3} T + \frac{1}{2} V(T) \mathcal{F}_{\mu_1 \mu_2} C \partial_{\mu_3} T \right) . \quad (102)$$

Now we proceed to the Hamiltonian formalism. The momenta conjugate to v, A_0 and ω^i are primary constraints

$$p_v \approx 0 , \tau_i \approx 0 , \pi^0 \approx 0 \quad (103)$$

while the momenta conjugate to $T X^M$ and A_i have the form

$$\begin{aligned}
p_T &= \frac{e^{-2\Phi} V^2}{v} \partial_\mu T (\tilde{\mathbf{A}}^{-1})_S^{\mu 0} \det \tilde{\mathbf{A}} - \frac{V}{v} (C_{12} + \mathcal{F}_{12} C) \star \tilde{\mathcal{G}} \\
p_M &= \frac{e^{-2\Phi} V^2}{v} \left(G_{MN} \partial_\mu X^N (\tilde{\mathbf{A}}^{-1})_S^{\mu 0} + B_{MN} \partial_i X^N (\tilde{\mathbf{A}}^{-1})_A^{i0} \right) \det \tilde{\mathbf{A}} \\
&\quad - \frac{\star \tilde{\mathcal{G}} V}{v} \epsilon^{i_1 i_2} (C_{MN} + B_{MN} C) \partial_{i_1} X^N \partial_{i_2} T , \\
\pi^i &= \frac{e^{-2\Phi} V^2}{v} (\tilde{\mathbf{A}}^{-1})_A^{i0} \det \tilde{\mathbf{A}} - \frac{\star \mathcal{G} V}{v} C \epsilon^{ij} \partial_j T , \quad \tau_0 = \frac{\star \tilde{\mathcal{G}}}{v} ,
\end{aligned} \tag{104}$$

where $\epsilon^{12} = -\epsilon^{21} = 1$. Following the same procedure as in previous sections we find the extended Hamiltonian in the form

$$H_E = \int d^2 \xi \left(\frac{v e^{2\Phi}}{2 \det \tilde{\mathbf{A}}_{ij} V^2} \tilde{\mathcal{H}}_0 + N^i \tilde{\mathcal{H}}_i + \omega^i \partial_i \tau_0 + N^i \tilde{\mathcal{H}}_i \right) , \tag{105}$$

where

$$\tilde{\mathcal{H}}_0 = \Pi_M G^{MN} \Pi_N + \Pi_T^2 + \Pi^i (\tilde{\mathbf{A}}_S)_{ij} \Pi^j + V^2 \tau_0^2 e^{-2\Phi} \det \tilde{\mathbf{A}}_{ij} , \tag{106}$$

where Π_M, Π_T and Π^j are defined as

$$\begin{aligned}
\Pi_T &= p_T + \tau_0 V (C_{12} + \mathcal{F}_{12}) \tau_0 , \\
\Pi_M &= p_M - B_{MN} \partial_i X^N \pi^i + \tau_0 V C_{MN} \partial_i X^N \epsilon^{ij} \partial_j T , \\
\Pi^i &= \pi^i + \tau_0 V C \epsilon^{ij} \partial_j T .
\end{aligned} \tag{107}$$

We could analyze this system in the same way as in previous sections. However we see from the form of the Hamiltonian density and from (107) that non-BPS D2-brane at the tachyon vacuum does not couple to the Ramond-Ramond fields and the dynamics of the configuration with the non-zero electric flux reduces to the dynamics of the Nambu-Gotto string in this background. This is a non-trivial result that supports the conjecture that the end point of the tachyon condensation on the world-volume of non-BPS Dp-brane is the gas of the tensile strings.

Acknowledgement:

This work was supported by the Grant Agency of the Czech Republic under the grant P201/12/G028.

References

- [1] R. Blumenhagen, D. Lüst and S. Theisen, “*Basic concepts of string theory*,” doi:10.1007/978-3-642-29497-6

- [2] K. Becker, M. Becker and J. H. Schwarz, “*String theory and M-theory: A modern introduction*,”
- [3] J. Polchinski, “*String theory. Vol. 2: Superstring theory and beyond*,”
- [4] E. Bergshoeff, L. A. J. London and P. K. Townsend, “*Space-time scale invariance and the superp-brane*,” *Class. Quant. Grav.* **9** (1992) 2545 doi:10.1088/0264-9381/9/12/002 [hep-th/9206026].
- [5] P. K. Townsend, “*Membrane tension and manifest IIB S duality*,” *Phys. Lett. B* **409** (1997) 131 doi:10.1016/S0370-2693(97)00862-9 [hep-th/9705160].
- [6] E. Bergshoeff and P. K. Townsend, “*Super D-branes revisited*,” *Nucl. Phys. B* **531** (1998) 226 doi:10.1016/S0550-3213(98)00432-5 [hep-th/9804011].
- [7] J. Polchinski, “*Dirichlet Branes and Ramond-Ramond charges*,” *Phys. Rev. Lett.* **75** (1995) 4724 doi:10.1103/PhysRevLett.75.4724 [hep-th/9510017].
- [8] R. Dijkgraaf, B. Heidenreich, P. Jefferson and C. Vafa, “*Negative Branes, Supergroups and the Signature of Spacetime*,” arXiv:1603.05665 [hep-th].
- [9] A. Sen, “*Supersymmetric world volume action for nonBPS D-branes*,” *JHEP* **9910** (1999) 008 doi:10.1088/1126-6708/1999/10/008 [hep-th/9909062].
- [10] E. A. Bergshoeff, M. de Roo, T. C. de Wit, E. Eyras and S. Panda, “*T duality and actions for nonBPS D-branes*,” *JHEP* **0005** (2000) 009 doi:10.1088/1126-6708/2000/05/009 [hep-th/0003221].
- [11] M. R. Garousi, “*Tachyon couplings on nonBPS D-branes and Dirac-Born-Infeld action*,” *Nucl. Phys. B* **584** (2000) 284 doi:10.1016/S0550-3213(00)00361-8 [hep-th/0003122].
- [12] J. Kluson, “*Proposal for nonBPS D-brane action*,” *Phys. Rev. D* **62** (2000) 126003 doi:10.1103/PhysRevD.62.126003 [hep-th/0004106].
- [13] A. Sen, “*Tachyon dynamics in open string theory*,” *Int. J. Mod. Phys. A* **20** (2005) 5513 doi:10.1142/S0217751X0502519X [hep-th/0410103].
- [14] A. Sen, “*Dirac-Born-Infeld action on the tachyon kink and vortex*,” *Phys. Rev. D* **68** (2003) 066008 doi:10.1103/PhysRevD.68.066008 [hep-th/0303057].
- [15] J. Kluson, “*Tachyon kink on non-BPS Dp-brane in the general background*,” *JHEP* **0510** (2005) 076 doi:10.1088/1126-6708/2005/10/076 [hep-th/0508239].
- [16] A. Sen, “*Open and closed strings from unstable D-branes*,” *Phys. Rev. D* **68** (2003) 106003 doi:10.1103/PhysRevD.68.106003 [hep-th/0305011].
- [17] O. K. Kwon and P. Yi, “*String fluid, tachyon matter, and domain walls*,” *JHEP* **0309** (2003) 003 doi:10.1088/1126-6708/2003/09/003 [hep-th/0305229].

- [18] A. Sen, “*Fundamental strings in open string theory at the tachyonic vacuum,*” J. Math. Phys. **42** (2001) 2844 doi:10.1063/1.1377037 [hep-th/0010240].
- [19] G. W. Gibbons, K. Hori and P. Yi, “*String fluid from unstable D-branes,*” Nucl. Phys. B **596** (2001) 136 doi:10.1016/S0550-3213(00)00716-1 [hep-th/0009061].
- [20] U. Lindstrom and M. Zabzine, “*Strings at the tachyonic vacuum,*” JHEP **0103** (2001) 014 doi:10.1088/1126-6708/2001/03/014 [hep-th/0101213].
- [21] U. Lindstrom, M. Zabzine and A. Zheltukhin, “*Limits of the D-brane action,*” JHEP **9912** (1999) 016 doi:10.1088/1126-6708/1999/12/016 [hep-th/9910159].
- [22] H. Gustafsson and U. Lindstrom, “*A Picture of D-branes at strong coupling. 2. Spinning partons,*” Phys. Lett. B **440** (1998) 43 doi:10.1016/S0370-2693(98)01080-6 [hep-th/9807064].
- [23] U. Lindstrom and R. von Unge, “*A Picture of D-branes at strong coupling,*” Phys. Lett. B **403** (1997) 233 doi:10.1016/S0370-2693(97)00548-0 [hep-th/9704051].
- [24] S. Hassani, U. Lindstrom and R. von Unge, “*Classically equivalent actions for tensionless p-branes,*” Class. Quant. Grav. **11** (1994) L79. doi:10.1088/0264-9381/11/5/002
- [25] K. Skenderis and M. Taylor, “*Branes in AdS and p p wave space-times,*” JHEP **0206** (2002) 025 doi:10.1088/1126-6708/2002/06/025 [hep-th/0204054].